Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, O.

Let l=the entire length of the chain, a=the radius of the wheel, x=the longest part hanging down, $l-(\pi a+x)$=the shortest part, θ=the angle which the radius through any point of the string makes with the vertical diameter and positive in the direction left to right, T and T' the tensions at any points in those parts of the string to which x and $l-(\pi a+x)$ belong, and μ=the coefficient of friction.

We now have, from the usual theory, a unit of length of the chain being taken as a unit of weight.

$$T= Ce^{\mu\theta}+\frac{a}{1+\mu^2}[2\mu\sin\theta+(1-\mu^2)\cos\theta]\ldots.(1).$$

When $\theta=\frac{1}{2}\pi$, $T=x$, and (1) is

$$x=Ca^{\frac{1}{2}(\mu\pi)}+\frac{2\mu a}{1+\mu^2}, \text{ or } C=\left(x-\frac{2\mu a}{1+\mu^2}\right)e^{-\frac{1}{2}(\mu\pi)}\ldots.(2),$$

and (1) is $T=\left(x-\dfrac{2\mu a}{1+\mu^2}\right)e^{\mu(\theta-\frac{1}{2}\pi)}+\dfrac{a}{1+\mu^2}[-2\mu\sin\theta+(1-\mu^2)\cos\theta]\ldots.(3).$

In like manner, $T'=l-(\pi a+x)$ when $\theta=-\frac{1}{2}\pi$, and

$$T'=\left(l-\pi a-x+\frac{2\mu a}{1+\mu^2}\right)e^{\mu(\theta+\frac{1}{2}\pi)}+\frac{a}{1+\mu^2}[-2\mu\sin\theta+(1-\mu^2)\cos\theta]\ldots.(4).$$

For equilibrium, $T=T$ at the vertex, where $\theta=0$.

$$\therefore \left(l-\pi a-x+\frac{2\mu a}{1+\mu^2}\right)e^{\frac{1}{2}(\mu\pi)}=\left(x-\frac{2\mu a}{1+\mu^2}\right)e^{-\frac{1}{2}(\mu\pi)}\ldots.(5),$$

giving $x=\dfrac{l-\pi a}{1+e^{-\mu\pi}}+\dfrac{2\mu a}{1+\mu^2}\ldots.(6).$

Then $2x+\pi a-l$ is found, the required length.

Also solved by *G. B. M. ZERR.*

AVERAGE AND PROBABILITY.

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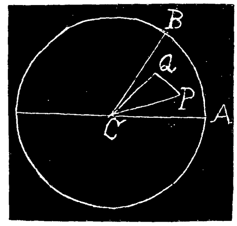
101. Proposed by L. C. WALKER, Assistant Professor of Mathematics, Leland Stanford Jr. University, Palo Alto, Cal.

By direct calculation obtain the average distance between two points in the surface of a circle.

I. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let P, Q be the random points in the sector AOB. $AO=r$, $OP=y$, $OQ=x$,

$\angle AOB=\beta$, $\angle AOQ=\theta$, $\angle AOP=\varphi$. An element of the sector at P is $ydyd\varphi$; at Q, $xdxd\theta$. The limits of x, are 0 and r; of y, 0 and x; of θ, 0 and β; of φ, 0 and θ.



$PQ=u=[x^2+y^2-2xy\cos(\theta-\varphi)]^{\frac{1}{2}}$.

$$\therefore \Delta=\frac{\int_0^\beta\int_0^\theta\int_0^r\int_0^x uxdxydyd\theta d\varphi}{\int_0^\beta\int_0^\theta\int_0^r\int_0^x xdxyd\theta d\varphi}$$

$$=\frac{16}{\beta^2r^4}\int_0^\beta\int_0^\theta\int_0^r\int_0^x uxyd\theta d\varphi dxdy=\frac{8}{3\beta^2r^4}\int_0^\beta\int_0^\theta\int_0^r\{16\sin^3\tfrac{1}{2}(\theta-\varphi)$$

$$+12\sin^3\tfrac{1}{2}(\theta-\varphi)\cos(\theta-\varphi)-2+3\cos^2(\theta-\varphi)$$

$$+3\sin^2(\theta-\varphi)\cos(\theta-\phi)\log[1+\operatorname{cosec}\tfrac{1}{2}(\theta-\varphi)]\}x^4d\theta d\varphi dx$$

$$=\frac{8r}{15\beta^2}\int_0^\beta\int_0^\theta\{16\sin^3\tfrac{1}{2}(\theta-\varphi)+12\sin^3\tfrac{1}{2}(\theta-\varphi)\cos(\theta-\varphi)-2+3\cos^2(\theta-\varphi)$$

$$+4\sin^2(\theta-\varphi)\cos(\theta-\varphi)\log[1+\operatorname{cosec}\tfrac{1}{2}(\theta-\varphi)]\}d\theta d\varphi$$

$$=\frac{4r}{45\beta^2}\int_0^\beta\{48\sin^4\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta+12\sin^3\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta-32\sin^2\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta-6\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta$$

$$-64\cos\tfrac{1}{2}\theta+64+9\sin\theta\cos\theta+6\sin^3\theta\log(1+\operatorname{cosec}\theta)\}d\theta$$

$$=\frac{2r}{135\beta^2}\{16\sin^6\tfrac{1}{2}\beta+16\cos^6\tfrac{1}{2}\beta+96\sin^5\tfrac{1}{2}\beta+24\sin^4\tfrac{1}{2}\beta-12\cos^4\tfrac{1}{2}\beta-112\sin^3\tfrac{1}{2}\beta$$

$$-36\sin^2\tfrac{1}{2}\beta-720\sin\tfrac{1}{2}\beta+27\sin^2\beta+384\beta-4$$

$$-12(\sin^2\beta\cos\beta+2\cos\beta+2)\log(1+\sin\tfrac{1}{2}\beta)+12(\sin^2\beta\cos\beta$$

$$+2\cos\beta-2)\log\sin\tfrac{1}{2}\beta\}.$$

For the circle, $\beta=2\pi$, $\Delta=\dfrac{128r}{45\pi}$.

For the semicircle, $\beta=\pi$, $\Delta=\dfrac{256r}{45\pi}-\dfrac{1472r}{135\pi^2}$.

For the quadrant, $\beta=\frac{1}{2}\pi$.

$$\Delta=\frac{32r}{135\pi}\left(48+\frac{3}{\pi}-\frac{94\sqrt{2}}{\pi}-\frac{6}{\pi}\log\frac{1+\sqrt{2}}{2}\right).$$

II. Solution by the PROPOSER.

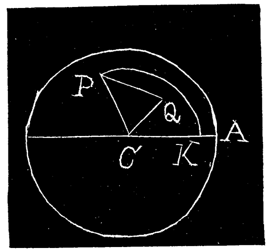
Take the center, C, of the circle and the horizontal radius CA as pole and initial line in polar coördinates.

Let $P(x, \theta)$ and $Q(y, \phi)$ be the random points. Then we have

$$PQ=\sqrt{x^2+y^2-2xy\cos(\theta-\phi)}.$$

Now with C as a center and a radius CP describe the arc PK.

It is evidently necessary to consider only those positions of the two points in which Q is confined to the sector PCK, since under this limitation PQ will take all possible distances.



An element of area of the circle at the point P is $xdxd\theta$, and at the point Q it is $ydyd\phi$.

The limits of y are 0 and x; of x, 0 and r; of ϕ, 0 and θ; and those of θ, 0 and 2π.

Consequently, we have for the required mean distance

$$M=\frac{1}{(\pi r^2)^2}\int_0^{2\pi}\int_0^{\theta}\int_0^{r}\int_0^{x}\sqrt{[x^2+y^2-2xy\cos(\theta-\phi)]}xdxd\theta.ydyd\phi$$

$$=\frac{4}{3\pi^2r^4}\int_0^{2\pi}\int_0^{\theta}\int_0^{x}\Big[8\sin^3\tfrac{1}{2}(\theta-\phi)+6\sin^3\tfrac{1}{2}(\theta-\phi)\cos(\theta-\phi)$$

$$-\tfrac{3}{2}\sin^2(\theta-\phi)\cos(\theta-\phi)\log\left(\frac{1+\sin\frac{1}{2}(\theta-\phi)}{\sin\frac{1}{2}(\theta-\phi)}\right)-1+\cos^2(\theta-\phi)\Big]x^4d\theta d\phi dx$$

$$=\frac{r}{15\pi^2}\int_0^{2\pi}\int_0^{\theta}\Big[8\sin\tfrac{1}{2}(\theta-\phi)+40\sin\tfrac{1}{2}(\theta-\phi)\cos^2\tfrac{1}{2}(\theta-\phi)$$

$$+6\sin^2(\theta-\phi)\cos(\theta-\phi)\log\left(\frac{1+\sin\frac{1}{2}(\theta-\phi)}{\sin\frac{1}{2}(\theta-\phi)}\right)$$

$$-48\sin\tfrac{1}{2}(\theta-\phi)\cos^4\tfrac{1}{2}(\theta-\phi)+3\cos2(\theta-\phi)-1\Big]d\theta d\phi$$

$$=\frac{2r}{45\pi^2}\int_0^{2\pi}\Big[32-32\cos\tfrac{1}{2}\theta-16\sin^2\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta+24\sin^4\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta+3\sin\theta\cos\theta$$

$$+3\sin^3\theta\log\left(\frac{1+\sin\frac{1}{2}\theta}{\sin\frac{1}{2}\theta}\right)\Big]d\theta=\frac{128r}{45\pi}.$$

Professor Walker also furnished a second very excellent solution.

MISCELLANEOUS.

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91. Proposed by ARTEMAS MARTIN, LL. D., U. S. Coast and Geodetic Survey Office, Washington, D. C.

The following sides and area are given for a rational triangle in the table of rational scalene triangles on page 167 of Dr. Halsted's "Metrical Geometry (Boston, 1881), viz.: sides, 21, 61, 65; area, 420. The same sides and area are given in Septimus Tebay's "Mensuration" (London and Cambridge, 1868), in a table on page 113.